State Space Introduction

MEM 355 Performance Enhancement of Dynamical Systems

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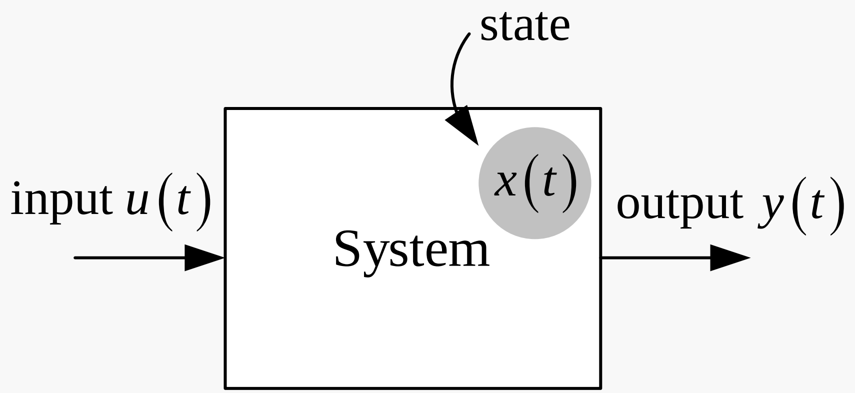
Outline

State space techniques emerged around 1960. They are direct and exploit the efficient computations of linear algebra.

- State space models
- The Resolvent
- Solving State Space Equations
- The Matrix Exponential
- Variation of Parameters Formula

State Space Models



The differential equation or 'state space' model is

$$\dot{x}(t) = Ax(t) + Bu(t) \quad \text{state equation}$$

$$y(t) = Cx(t) + Du(t) \quad \text{output equation}$$

$$x(0) = x_0 \quad \text{initial condition}$$

The state space model describes how the input $u(t)$ and the initial condition affect the state $x(t)$ and the output $y(t)$.

Solving State Equations via the Laplace Transform

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

$$\begin{aligned} \mathscr{L}(\dot{x}) &= A\mathscr{L}(x) + B\mathscr{L}(u) \\ \mathscr{L}(y) &= C\mathscr{L}(x) + D\mathscr{L}(u) \end{aligned} \quad \Rightarrow \quad \begin{aligned} sX(s) - x_0 &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s) \end{aligned}$$

$$X(s) = [sI - A]^{-1} x_0 + [sI - A]^{-1} BU(s)$$

$$Y(s) = C[sI - A]^{-1} x_0 + \left\{C[sI - A]^{-1} B + D\right\} U(s)$$

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The Resolvent

$$[sI - A]^{-1} = \frac{\text{adj}(sI - A)}{\det(sI - A)} \approx \frac{n \times n \text{ matrix}}{\det(sI - A)}$$

$$\text{adj}(sI - A) = n \times n \text{ matrix of cofactors}$$

Recall, the n^2 minors of an $n \times n$ matrix M are defined as: the i, j minor M_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix obtained from M by deleting the i^{th} row and j^{th} column.

The i, j cofactor is $C_{ij} = (\text{-}1)^{i+j} M_{ij}$

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Solving Linear State Equations

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x \in R^n, u \in R^m$$

$$given: x(t_0) = x_0, u(t) \ for \ t \geq t_0$$

$$find: x(t) \ for \ t \geq t_0$$

$$\dot{x} = Ax + b(t), \quad b(t) := Bu(t), \text{ forced or nonhomogeneous}$$

$$\dot{x} = Ax \qquad \text{homogeneous}$$

Solution Strategy:

1) find general sol'n to homogeneous eq. -
   will involve $n = \dim x$ arbitrary parameters
2) find any particular solution
3) add and pick parameters to match initial condition

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Solution in the Time Domain

$$x_1(t), x_2(t) \quad \text{sol'ns of homog.}, c_1, c_2 \text{ constants}$$

$$\Downarrow$$

$$x(t) = c_1 x_1(t) + c_2 x_2(t) \text{ is a sol'n of homog.}$$

$$x_1(t), x_2(t) \quad \text{sol'ns of forced}$$

$$\Downarrow$$

$$x(t) = x_1(t) - x_2(t) \text{ is a sol'n of homog.}$$

$$x_p(t) \text{ any sol'n of forced, } x_h(t) \text{ any sol'n of homog.}$$

$$\Downarrow$$

$$x(t) = x_h(t) + x_p(t) \text{ is a sol'n of forced}$$

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The Homogeneous Equation

Let us first solve the homogeneous equation

$$\dot{x}(t) = Ax(t),\ x(t_0) = x_0$$

Strategy: assume a sol'n and see if it works.

Assume a solution in the form of a power series:

$$x(t) = \mathbf{a}_0 + \mathbf{a}_1(t - t_0) + \mathbf{a}_2(t - t_0)^2 + \cdots + \mathbf{a}_k(t - t_0)^k + \cdots$$

$$\dot{x}(t) \Rightarrow \mathbf{a}_1 + 2\mathbf{a}_2(t - t_0) + \cdots + k\mathbf{a}_k(t - t_0)^{k-1} + \cdots$$

$$Ax(t) \Rightarrow A\mathbf{a}_0 + A\mathbf{a}_1(t - t_0) + A\mathbf{a}_2(t - t_0)^2 + \cdots + A\mathbf{a}_k(t - t_0)^k + \cdots$$

Compare coefficients of like powers of $(t - t_0)$

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The Homogeneous Equations, 2

$$\begin{aligned} \mathbf{a}_1 &= A\mathbf{a}_0 & \mathbf{a}_1 &= A\mathbf{a}_0 \\ \mathbf{a}_2 &= \frac{1}{2}A\mathbf{a}_1 & \mathbf{a}_2 &= \frac{1}{2}A^2\mathbf{a}_0 \\ &\vdots \quad \Rightarrow & &\vdots \\ \mathbf{a}_k &= \frac{1}{k}A\mathbf{a}_{k-1} & \mathbf{a}_k &= \frac{1}{k!}A^k\mathbf{a}_0 \\ &\vdots & &\vdots \end{aligned}$$

$$\begin{aligned} x(t) &= \mathbf{a}_0 + A\mathbf{a}_0(t-t_0) + \frac{1}{2}A^2\mathbf{a}_0(t-t_0)^2 + \cdots + \frac{1}{k!}A^k\mathbf{a}_0(t-t_0)^k + \cdots \\ &= \left[I + A(t-t_0) + \frac{1}{2}A^2(t-t_0)^2 + \cdots + \frac{1}{k!}A^k(t-t_0)^k + \cdots\right]\mathbf{a}_0 \end{aligned}$$

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The Homogeneous Equation, 3

Set $t = t_0,\ x(t_0) = x_0$ to obtain

$$\mathbf{a}_0 = x_0 \Rightarrow$$

$$x(t) = \left[I + A(t - t_0) + \frac{1}{2} A^2 (t - t_0)^2 + \cdots + \frac{1}{k!} A^k (t - t_0)^k + \cdots \right] x_0$$

Recall the series expansion for the (scalar) exponential

$$e^{a(t-t_0)} = 1 + a(t - t_0) + \frac{1}{2} a^2 (t - t_0)^2 + \cdots + \frac{1}{k!} a^k (t - t_0)^k + \cdots$$

Define the matrix exponential

$$e^{A(t-t_0)} \triangleq I + A(t - t_0) + \frac{1}{2} A^2 (t - t_0)^2 + \cdots + \frac{1}{k!} A^k (t - t_0)^k + \cdots$$

so that

State transition matrix

$$x(t) = e^{A(t-t_0)} x_0 = \Phi(t, t_0)\, x_0$$

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Matrix Exponential

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \cdots + \frac{1}{k!}A^kt^k + \cdots$$

Some properties:

$$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$$

$$e^{At}e^{-At} = I \Rightarrow \left[e^{At}\right]^{-1} = e^{-At}$$

$$e^{At}e^{Bt} = e^{(A+B)t} \text{ if and only if } AB = BA$$

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Variation of Parameters Formula

Recall, any sol'n of (forced) satisfies

$$x(t) = x_h(t) + x_p(t)$$

where

$x_h(t) = e^{At}c$ for constant vector c, satisfies (homog.)

$x_p(t)$ is any (particular) sol'n of (forced)

We seek $x_p(t)$.

Assume the form $x_p(t) = e^{At}c(t)$.

Variation of Parameters, 2

$$\dot{x}_p = Ax_p + Bu \text{ and } \frac{d}{dt}e^{At}c(t) = Ae^{At}c(t) + e^{At}\dot{c}(t)$$

$$\Rightarrow \dot{c}(t) = e^{-At}B(t)u(t)$$

$$\Rightarrow c(t) = \cancel{c(t_0)} + \int_{t_0}^{t} e^{-A\tau}B(\tau)u(\tau)\,d\tau$$

Now,

$$x(t) = e^{At}c + e^{At}\int_{t_0}^{t} e^{-A\tau}B(\tau)u(\tau)\,d\tau$$

$$= e^{At}c + \int_{t_0}^{t} e^{A(t-\tau)}B(\tau)u(\tau)\,d\tau$$

Variation of Parameters, 3

Finally, $x(t_0) = x_0 \Rightarrow x_0 = e^{At_0} c \Rightarrow c = e^{-At_0} x_0$

$$\boxed{\begin{aligned} x(t) &= e^{A(t-t_0)} x_0 + \int_{t_0}^{t} e^{A(t-\tau)} B(\tau) u(\tau)\, d\tau \\ &= \Phi(t,t_0) x_0 + \int_{t_0}^{t} \Phi(t,\tau) B(\tau) u(\tau)\, d\tau \end{aligned}}$$

Recall (with t_0=0)

$$X(s) = [sI - A]^{-1} x_0 + [sI - A]^{-1} BU(s)$$

By comparison,

$$\boxed{\mathscr{L}\left[e^{At}\right] = [sI - A]^{-1}}$$

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Example - MATLAB

```
>> A=[1 0 0;0 2 1;2 0 0];
>> syms t
>> expm(t*A)

[                exp(t),                        0,                     0]
[ exp(2*t)-2*exp(t)+1,                   exp(2*t),   -1/2+1/2*exp(2*t)]
[            2*exp(t)-2,                        0,                     1]
>> laplace(expm(t*A))

[              1/(s-1),                        0,                     0]
[ 1/(s-2)-2/(s-1)+1/s,                    1/(s-2),    -1/2/s+1/2/(s-2)]
[           2/(s-1)-2/s,                        0,                   1/s]
```

Summary

- State transition matrix
- Matrix exponential
- Resolvent
- Variation of parameters formula